

Solutions of the semiclassical Einstein's equations with applications in cosmology²

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Summary

- ▶ Cosmological Scenario
- ▶ Semiclassical Einstein's equation
- ▶ Stress-Energy Tensor regularization
- ▶ Solution with scalar conformal fields as sources
- ▶ Solution with massive fields as sources

Cosmological scenario: geometry

- Physical input: Universe is homogeneous and isotropic.
Then FRW metric:

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 + \kappa r^2} + r^2 d\Sigma^2 \right).$$

$\kappa = 0$ flat, $\kappa = \pm 1$ open or closed.

- recent observation: $a(t) \simeq Ce^{Ht}$, and $\kappa \simeq 0$.

Solutions that have the FRW form.

Cosmological scenario: matter

- ▶ We model T_{ab} as a perfect fluid

$$T_{ab} = \rho u_a u_b + P(g_{ab} + u_a u_b).$$

Homogeneity and isotropy $\implies u = \frac{\partial}{\partial t}$, $\rho(t)$ and $P(t)$

- ▶ Einstein's equations become FRW equations $H = \dot{a}a^{-1}$

$$3H^2 = 8\pi\rho - \frac{3\kappa}{a^2} \quad (1)$$

$$3\dot{H} + 3H^2 = -4\pi(\rho + 3P) \quad (2)$$

- ▶ Type of fluids: $P = w\rho$ and conservation equation

- ▶ Radiation: $w = \frac{1}{3}, \quad \rho_R \sim a(t)^{-4}$
- ▶ Dust: $w = 0, \quad \rho_M \sim a(t)^{-3}$
- ▶ Cosmological constant: $w = -1 \quad \rho_\Lambda = C$

Cosmological scenario: observation

- ▶ If we use this to model the present day observation:
 - ▶ Radiation is not important.
 - ▶ We look for a mixture of ρ_M and ρ_Λ
- ▶ To model CMB and Supernovae red-shift observation:

We have a problem

At the present time:

Energy density: $\sim 70\%$ *Dark Energy*, $\sim 30\%$ *Matter*.

Known matter: only $\sim 4\%$.

- ▶ Let's try to see the role of quantum effects.

Gravity: semiclassical approximation

- ▶ We would like to have a quantum theory of gravity.
- ▶ **Too difficult.**
- ▶ At least we would like to have a theory of backreaction.
- ▶ We try semiclassically.

$$G_{ab} = 8\pi \langle T_{ab} \rangle.$$

- ▶ It should work in some regimes. As in atomic physics: quantum mechanical electron with external classical field.

Range of validity of semiclassical approximation

- ▶ A complete satisfactory semiclassical description is impossible. (quantum matter is a source for gravity).
- ▶ It should be valid whenever quantum fluctuations are negligible.
- ▶ In some models, backreaction is unavoidable: Particle creation.
(Ex: black holes radiates)
 - ▶ Are there quantum effects that can be seen?
 - ▶ How is modified the vacuum energy?
 - ▶ How can be treated the backreaction effects?
 - ▶ Are they a small effect?
 - ▶ What implication has the quantum origin of matter on the solutions?

Wald Axioms

In QM T_{ab} are singular objects $\langle T_{ab} \rangle \rightarrow \infty$.

We need a renormalization prescription for T_{ab} on CST.

Wald axioms $\implies$ meaningful semiclassical approx.

[Wald 77] [Wald 78]

- (1.) It must agree with formal results for T_{ab}
(For scalar: $(\Phi, T_{ab}\Psi)$, can be found formally if $(\Phi, \Psi) = 0$).
- (2.) Regularization of T_{ab} in Minkowski coincide with “normal ordering”.
- (3.) Conservation: $\nabla^a \langle T_{ab} \rangle = 0$.
- (4.) Causality: $\langle T_{ab} \rangle$ at p depends only on $J^-(p)$.
- (5.) T_{ab} depends on derivatives of the metric up to the second order (or third).

Nice Environment

Problem:

How can we treat matter without fixing the spacetime?

- ▶ We can quantize simultaneously and coherently on all spacetimes. *[Brunetti Fredenhagen Verch 2003]*.
- ▶ Quantum Fields are particular observables that transform suitably under isomorphisms.
- ▶ We need another ingredient:
 “reference states” on every spacetime.
- ▶ Einstein's eq.
 - ▶ Consistency criterion.
 - ▶ Selects particular elements in the category of local Manifolds.

What we need to search for in a quantum theory

Instead of considering FRW equation we use the following.

$$-R = 8\pi T, \quad \nabla^a T_{ab} = 0$$

- ▶ Up to some initial condition
(it remains the freedom of fixing $a(t_0) = a_0$).
- ▶ But it is simpler to perform quantum computations.

Matter: Scalar free field theory

Equation of motion

$$P := -\square + \xi R + V, \quad P\phi = 0.$$

We will be interested in the case $V = m^2$ and $\xi = 1/6$.

Stress-Energy Tensor:

$$\begin{aligned} T_{ab} := & \partial_a \phi \partial_b \phi - \frac{1}{6} g_{ab} (\partial_c \phi \partial^c \phi + V \phi^2) - \xi \nabla_{(a} \partial_{b)} \phi^2 \\ & + \xi \left(R_{ab} - \frac{R}{6} g_{ab} \right) \phi^2 + \left(\xi - \frac{1}{6} \right) g_{ab} \square \phi^2. \end{aligned}$$

It differs by the usual one by terms of the form $\phi P \phi$ [\[Moretti 2003\]](#).

Remarks on T_{ab}

- Trace

$$T = -3 \left(\frac{1}{6} - \xi \right) \square \phi^2 - V \phi^2 ,$$

where we have used $\phi P \phi = 0$ to simplify.

- Conservation equation

$$\nabla_a T^a_b = -\frac{1}{2} \phi^2 \partial_b V .$$

- Classical Ambiguity: $\phi P \phi = 0$

$$T'_{ab} = T_{ab} + C g_{ab} (\phi P \phi + P \phi \phi) .$$

- T_{ab} can be written by means of balanced derivatives and derivatives of the field ϕ^2 [*Buchholz Ojima Roos 2002*].

Quantum field theory

States become distribution,

- ▶ QFT described by n -point functions.
- ▶ Quasi free states ω described by the two-points function

$$\omega_2(x, y) = \langle \phi(x)\phi(y) \rangle$$

thought as distribution in $\mathcal{D}'(M \times M)$.

- ▶ T_{ab} arises as an operation on ω_2 and a coinciding point limit.
- ▶ It is not well defined...
- ▶ Quasifree states that possess Hadamard property

[Radzikowski 1995] [Brunetti Fredenhagen Köhler 1996]

$$\text{WF}(\omega_2) = \{((x_1, k_1), (x_2, k_2)) \in T^*M^2 / \{0\} : -P_\gamma k_2 = k_1 > 0\}$$

- ▶ **Physically:** The fluctuations of the field are always finite on Hadamard states.

Hadamard Two-points function

$$\omega_2 = \frac{1}{8\pi^2} \left(\frac{u}{\sigma_\epsilon} + v \log \sigma_\epsilon + w \right).$$

- ▶ u v w are smooth functions,
- ▶ u depends only upon the geometry via g_{ab}
- ▶ v depends upon g_{ab} , ξ and V
- ▶ w characterizes the state.

Some notations: σ is half of the square of the geodesic distance

$$v = \sum_{n=0}^{\infty} v_n \sigma^n \qquad [v](x) = v(x, x)$$

The singular Structure H is fixed and does not depend on the state.

Regularization of the two-points function

Regularization with point splitting: Minimal requirement.

$$\langle \phi(x)\phi(y) \rangle_\omega := \omega_2(x, y) - H(x, y)$$

It reduces to normal ordering for flat spacetime.

T_{ab} build on it. *[Hollands Wald, Brunetti Fredenhagen Verch, Moretti]*

$$8\pi^2 \langle \phi P \phi \rangle_\omega = 6[v_1], \quad 8\pi^2 \langle (\nabla_a \phi)(P\phi) \rangle_\omega = 2\nabla_a[v_1]$$

Conservation equation for T_{ab} are satisfied quantum mechanically

$$\nabla_a \langle T^a_b \rangle_\omega = -\frac{1}{2} \langle \phi^2 \rangle_\omega \partial_b V = -\frac{1}{2} \frac{[w]}{8\pi^2} \partial_b V$$

but unfortunately the trace is different from the classical one.

$$\langle T \rangle_\omega := \frac{2[v_1]}{8\pi^2} + \left(-3 \left(\frac{1}{6} - \xi \right) \square - V \right) \frac{[w]}{8\pi^2}.$$

Some computations.....

with $V = m^2$

$$\begin{aligned}
 2[v_1] = & \frac{1}{360} \left(C_{ijkl} C^{ijkl} + R_{ij} R^{ij} - \frac{R^2}{3} + \square R \right) + \frac{1}{4} \left(\frac{1}{6} - \xi \right)^2 R^2 + \\
 & + \frac{m^4}{4} - \frac{1}{2} \left(\frac{1}{6} - \xi \right) m^2 R + \frac{1}{12} \left(\frac{1}{6} - \xi \right) \square R.
 \end{aligned}$$

Remaining freedom

In the trace $c\Box R$. Wald's fifth axiom does not hold!

- ▶ We can add conserved tensors t_{ab} build out of curvature only.
- ▶ It must behave as T_{ab} under “scale” transformations.
- ▶ Some possibilities arises from the variation of

$$t_{ab} = \frac{\delta}{\delta g^{ab}} C \int \sqrt{g} R^2 + D \int \sqrt{g} R_{ab} R^{ab}$$

- ▶ $t_a{}^a = \alpha \Box R$
- ▶ We use this freedom to cancel the $\Box R$ term from $\langle T \rangle$.

Wald's fifth axiom partially holds for $\langle T'_{ab} \rangle = \langle T_{ab} \rangle - ct_{ab}$

$f(R)$ gravity

NB: t_{ab} alone does not guaranty stable solutions.

[Cognola Elizalde Odintsov Zerbini 05, Cognola Zerbini 06]

Equation of the universe

Assuming $\kappa = 0$, we write the equation $-R = 8\pi\langle T \rangle$ as follows

$$\begin{aligned}
 -6\left(\dot{H} + 2H^2\right) &= 8\pi G \left(-3\left(\frac{1}{6} - \xi\right)\square - m^2\right)\langle\phi^2\rangle_\omega + \\
 &+ \frac{G}{\pi} \left(-\frac{1}{30}\left(\dot{H}H^2 + H^4\right) + 9\left(\frac{1}{6} - \xi\right)^2\left(\dot{H}^2 + 4H^2\dot{H} + 4H^4\right)\right) \\
 &+ \frac{G}{\pi} \left(\frac{m^4}{4} - 3\left(\frac{1}{6} - \xi\right)m^2\left(\dot{H} + 2H^2\right)\right)
 \end{aligned}$$

If $\xi = 1/6$, namely for the conformal coupling it simplifies a lot:

$$-6\left(\dot{H} + 2H^2\right) = -8\pi G m^2\langle\phi^2\rangle_\omega + \frac{G}{\pi} \left(-\frac{1}{30}\left(\dot{H}H^2 + H^4\right) + \frac{m^4}{4}\right)$$

Conformal invariant theory

If $\xi = \frac{1}{6}$, $m^2 = 0$, the equation does not depend on the state.

$$\dot{H} \left(H^2 - \frac{H_c^2}{2} \right) = -H^4 + H_c^2 H^2, \quad H_c^2 = \frac{360\pi}{G}$$

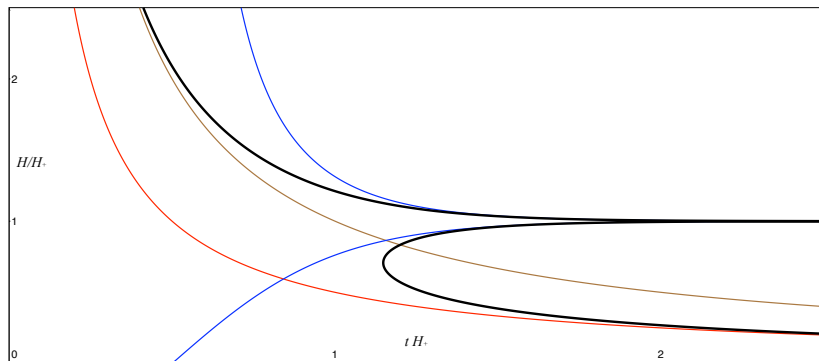
$H^2 = H_c^2$ and $H^2 = 0$ are solutions (*de Sitter, Minkowski*).

They are both stable as seen by the full solution

$$Ce^{4t} = e^{2/H} \left| \frac{H + H_c}{H - H_c} \right|^{1/H_c}$$

- It is as in the Starobinsky model but now with stable de Sitter. [*Starobinsky 80, Vilenkin 85*]

Clearly $H = 0$ and $H = H_c = H_+$ are stable solutions.



- ▶ $H = H_c$ is order of magnitude too big to describe the present expansion velocity of the universe.
- ▶ Two fixed points instead of one, a length scale is introduced (proportional to G).

Particle horizon

$\mathbb{R}^+ \times \Sigma$, in t_0 singularity.

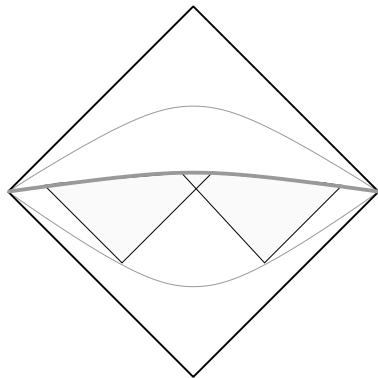
$$\tau = \int_t^{t_1} \frac{dt}{a(t)}$$

Where is $\tau(t_0)$?

$$ds^2 = -dt^2 + a^2 d\mathbf{x}^2$$

$$ds^2 = a^2 (-d\tau^2 + d\mathbf{x}^2)$$

Maximal **comoving distance** (if $c = 1$) it is τ .



- ▶ Radiation dominated:
 $\tau = \tau_0 - A(t - t_0)^{1/2} \rightarrow \tau_0$
 for $t \rightarrow t_0$
- ▶ Matter dominated:
 $\tau = \tau_0 - A(t - t_0)^{1/3} \rightarrow \tau_0$
 for $t \rightarrow t_0$
- ▶ $\rho = 1/a(t)^2$
 $\tau = \tau_0 - \log(t - t_0) \rightarrow -\infty$
 for $t \rightarrow t_0$

Massive model

Important: The quantum states enter in the equation.

$$\langle \phi^2 \rangle = \frac{[w]}{8\pi^2} + Am^2 + BR. \quad \text{We assume } [w] = 0.$$

$$\dot{H} (H^2 - H_0^2) = -H^4 + 2H_0^2 H^2 + M$$

where H_0 and M are two constants with the following values

$$H_0^2 = \frac{180\pi}{G} - B, \quad M = \frac{15}{2}m^4 - 240\pi^2 m^4 A$$

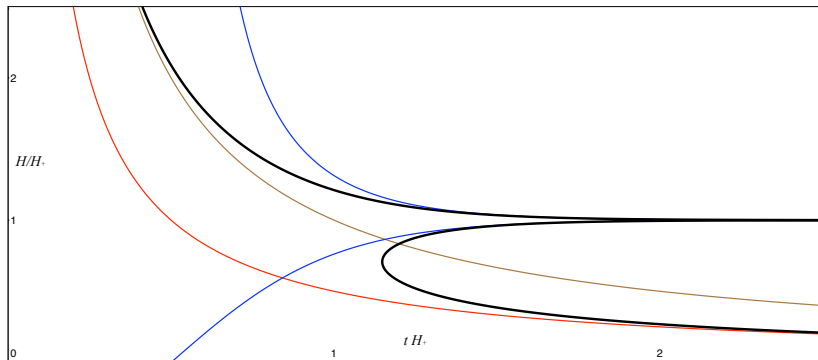
At most two fixed point (de Sitter phases)

$$H_{\pm}^2 = H_0^2 \pm \sqrt{H_0^4 + M},$$

they appear to be both stable.

We want to have Minkowski $H_- = 0$, $\implies A = (32\pi^2)^{-1}$.

Freedom in m and B to “Fine tune” H_+ .



- ▶ H_+ can be made small by suitable choices of m^2 and A, B
- ▶ It could model dark energy.
- ▶ Quantum effects are hardly negligible.
- ▶ Smooth exit from rapid expansion in the past. [*Shapiro Sola 02*]

Massive models and adiabatic vacuum $\langle \phi^2 \rangle$

On the Bunch-Davies state in dS $\langle \phi^2 \rangle$ is a constant.

We would like to choose “**ground states**”.

Impossible. Adiabatic states, have similar properties.

[Parker, Parker and Fulling, Lüders Roberts, Junker Schrohe, Olbermann]

- ▶ Minimize the particle creation rate. *[Parker]*
- ▶ Are states of minimal energy in the sense of Fewster.
[Olbermann]
- ▶ They can be thought as approximated ground states.

Let's see how they are constructed.

Two-points function

We consider $\xi = 1/6$, we use the conformal time:

$$\tau = \int \frac{dt}{a(t)}, \quad f'(\tau) = a(\tau)\dot{f}(\tau)$$

Two-points function:

$$\omega(x_1, x_2) = \frac{1}{8\pi^3} \frac{1}{a(\tau_1)a(\tau_2)} \int d^3\mathbf{k} \overline{T_k(\tau_1)} T_k(\tau_2) e^{i\mathbf{k} \cdot (\mathbf{x}_1 - \mathbf{x}_2)},$$

with

$$T_k'' + k^2 T_k + m^2 a(\tau)^2 T_k = 0, \quad \overline{T_k} T_k' - T_k \overline{T_k}' = i.$$

WKB approximation:

$$T_k(\tau) = \frac{1}{\sqrt{2\Omega_k(\tau)}} e^{i \int_{\tau_0}^{\tau} \Omega_k(\tau') d\tau'}.$$

Recursive relations

Then Ω_k^2 must satisfy the following equation

$$\Omega_k^2 = k^2 + m^2 a(\tau)^2 + \frac{5}{16} \left(\frac{(\Omega_k^2)'}{\Omega_k^2} \right)^2 - \frac{1}{4} \frac{(\Omega_k^2)''}{\Omega_k^2}$$

Recursively $\Omega_k^{(0)2} = k^2 + m^2 a(\tau)^2$ and $\Omega_k^{(n+1)}$ plugging $\Omega_k^{(n)}$ on the right.

The approx. two-point function is $\omega_2^{(n)}(x, y)$ is found using $\Omega_k^{(n)}$.

$$\langle \phi^2 \rangle_{(n)}(x) = \lim_{y \rightarrow x} (\omega_2^{(n)}(x, y) - H(x, y)) + \alpha'' R + \beta'' m^2$$

$\omega^{(n)}$ becomes closer to an Hadamard state $n \rightarrow \infty$.

The expectation value of $\langle \phi^2 \rangle$

If $\Omega^{(n)2} \geq 0$

$$\langle \phi^2 \rangle_{(n)} = \frac{1}{4\pi^2 a(\tau)^2} \int_0^\infty dk k^2 \left(\frac{1}{\Omega_k^{(n)}(\tau)} - \frac{1}{\Omega_k^{(0)}(\tau)} \right) + \alpha' R + \beta' m^2.$$

Problem: we have a good control only in the $k \gg 1$.

We expand the integral in powers of $1/m^2$

$$\langle \phi^2 \rangle_{(n)} = \alpha m^2 + \beta R + O\left(\frac{1}{m^2}\right)$$

The regime $m^2 \gg R$ is what we need. If $m = 1 \text{ GeV}$ $\frac{m^2}{R} \sim 10^{82}$

Summary

- ▶ Semiclassical solutions of Einstein's equation.
- ▶ The solutions depend upon the quantum states.
- ▶ The de Sitter phases could be stable only fixing the renormalization freedom.
- ▶ That solution shows a phase of rapid expansion at the beginning.

Open Questions

- ▶ How can we choose a nicer state?
- ▶ Fluctuations?
- ▶ Connection with $f(R)$ gravity?
- ▶ Origin of R^2 terms in the action? An hint on quantum gravity?